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Alpha-spiral mappings of a Banach space into the complex plane

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Abstract

Let E be a complex Banach space and let B be the unit ball in E , i.e. $B = \{x \in E : \|x\| < 1\}$. In this paper we define the class of α -spiral mappings of the unit ball B into the complex plane \mathbb{C} .

1 Introduction

Let E^* be the dual space of E . For any $A \in E^*$ we consider $\chi(A) = \{x \in E : A(x) \neq 0\}$ and $\gamma(A) = E \setminus \chi(A)$. If $A \neq 0$ then $\chi(A)$ is dense in E and $\chi(A) \cap \hat{B}$ is dense in \hat{B} , where $\hat{B} = \{x \in E : \|x\| = 1\}$.

Let $H(B)$ be the family of all functions $f : B \rightarrow \mathbb{C}$, $f(0) = 0$, which are holomorphic in B , i.e. have the Fréchet derivative $f'(x)$ in each point $x \in B$. If $f \in H(B)$, then, in some neighbourhoods V of the origin, $f(x) = \sum_{m=1}^{\infty} P_{m,f}(x)$, where the series is uniformly convergent on V and $P_{m,f} : E \rightarrow \mathbb{C}$ are continuous and homogeneous polynomials of degree m .

Let $\alpha \in \mathbb{R}$ with $|\alpha| < \frac{\pi}{2}$ and let $z_0 \in \mathbb{C} \setminus \{0\}$. The condition

$$z(t) = z_0 e^{-(\cos \alpha + i \sin \alpha)t}, \quad t \in \mathbb{R}$$

defines an α -spiral curve in the complex plane.

Let D be a domain in \mathbb{C} , such that $0 \in D$. If for any $z_0 \in D \setminus \{0\}$ the arc of α -spiral curve between the points z_0 and the origin is contained in D , then D is an α -spiral domain with respect to the origin.

Let $U = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $SP(\alpha)$ the family of all univalent functions $f : U \rightarrow \mathbb{C}$,

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are α -spiral in U , i.e. $f(U)$ is an α -spiral domain with respect to the origin.

Theorem 1 ([3]) *Let f be an holomorphic function from U into \mathbb{C} such that $f(0) = 0$, $f'(0) = 1$, $f(z) \neq 0$, for all $z \in U \setminus \{0\}$ and let $\alpha \in \mathbb{R}$ with $|\alpha| < \frac{\pi}{2}$. Then $f \in SP(\alpha)$ if and only if*

$$\operatorname{Re} \left[e^{i\alpha} \frac{zf'(z)}{f(z)} \right] > 0 \quad \text{for all } z \in U.$$

2 The class of alpha-spiral mappings on a Banach space

Let $A \in E^*$, $A \neq 0$ and $\alpha \in \mathbb{R}$, $|\alpha| < \frac{\pi}{2}$. We denote by $SP_A(\alpha)$ the family of all functions $f \in H(B)$ which have the form

$$f(x) = A(x) + \sum_{n=2}^{\infty} P_{n,f}(x) \quad (1)$$

such that, for any $a \in \chi(A) \cap \hat{B}$, f is univalent on $B_a = \{za : z \in U\}$ and $f(B_a)$ is an α -spiral domain with respect to the origin.

For any function f of the form (1) and $a \in \chi(A) \cap \hat{B}$ we consider the function $f_a : U \rightarrow \mathbb{C}$.

$$f_a(z) = \frac{f(za)}{A(a)}, \quad z \in U.$$

Obviously

$$f_a(z) = z + \sum_{n=2}^{\infty} \frac{P_{n,f}(a)}{A(a)} z^n, \quad z \in U. \quad (2)$$

Moreover, it is easy to check that

$$f_a^{(n)}(z) = \frac{f^{(n)}(za)(a, \dots, a)}{A(a)}, \quad n \in \mathbb{N}, z \in U.$$

By using the properties of α -spiral functions in the unit disk, we obtain some estimations of $|P_{n,f}(a)|$ and $\|P_{n,f}\|$ in the class $SP_A(\alpha)$.

Theorem 2 *If $f \in SP_A(\alpha)$ and $a \in \hat{B}$, then*

$$|P_{n,f}(a)| \leq \frac{|A(a)|}{(n-1)!} \prod_{k=1}^{n-1} [(k-1)^2 + 4k \cos^2 \alpha]^{\frac{1}{2}}, \quad n \geq 2 \quad (3)$$

This inequality is sharp and the equality holds for the function

$$f(x) = \frac{A(x)}{(1 - H(x))^{2s}}, \quad x \in B$$

where $s = e^{-i\alpha} \cos \alpha$, $H \in E^*$, $H(a) = 1$ and $\|H\| = 1$.

Proof. Suppose that $f \in SP_A(\alpha)$ and $n \geq 2$. If $a \in \chi(A) \cap \hat{B}$, then $f_a \in SP(\alpha)$ and hence we get (3). If $a \in \gamma(A) \cap \hat{B}$, then evidently $a = \lim_{m \rightarrow \infty} a_m$, where $a_m \in \chi(A)$, $m \in \mathbb{N}$. There exists $r_m \in \mathbb{R}_+$ such that $a_m/r_m \in \hat{B}$. Clearly $(r_m)_{m \geq 0}$ is bounded for 0 is an interior point of B .

Since $a_m/r_m \in \chi(A) \cap \hat{B}$, $m \in \mathbb{N}$, by the first part of the proof we have

$$\left| P_{n,f} \left(\frac{a_m}{r_m} \right) \right| \leq \left| A \left(\frac{a_m}{r_m} \right) \right| \frac{1}{(n-1)!} \prod_{k=1}^{n-1} [(k-1)^2 + 4k \cos^2 \alpha]^{\frac{1}{2}}, \quad m \in \mathbb{N}.$$

Hence

$$|P_{n,f}(a_m)| \leq r_m^{n-1} \frac{|A(a_m)|}{(n-1)!} \prod_{k=1}^{n-1} [(k-1)^2 + 4k \cos^2 \alpha]^{\frac{1}{2}}, \quad m \in \mathbb{N}.$$

By taking the limit with $m \rightarrow \infty$, we obtain $P_{n,f}(a) = 0$.

Corollary 1

Any $f \in SP_A(\alpha)$ vanishes on $\gamma(A) \cap B$.

Proof. Let $f \in SP_A(\alpha)$. Since $P_{n,f}(a) = 0$ for all $a \in \gamma(A) \cap \hat{B}$, f vanishes on B_a . Let $x \in \gamma(A) \cap B, x \neq 0$. Then $a = \frac{x}{\|x\|} \in \gamma(A) \cap \hat{B}$ and $f(za) = 0$ for all $z \in U$. Putting $z = \|x\|$, we get $f(x) = 0$.

Corollary 2 If $f \in SP_A(\alpha)$ and $n \geq 2$, then

$$\|P_{n,f}\| \leq \frac{\|A\|}{(n-1)!} \prod_{k=1}^{n-1} [(k-1)^2 + 4k \cos^2 \alpha]^{\frac{1}{2}} \quad (4)$$

The inequality is sharp, being attained by

$$f(x) = \frac{A(x)}{(1 - H(x))^{2s}}, \quad x \in B.$$

We shall give some necessary and sufficient conditions for holomorphic functions to belong to the class $SP_A(\alpha)$.

Theorem 3 *If $f \in SP_A(\alpha)$, then*

$$\operatorname{Re} \left[e^{i\alpha} \frac{f'(x)(x)}{f(x)} \right] > 0, \quad \text{for any } x \in \chi(A) \cap B \quad (5)$$

Proof. Let $x \in \chi(A) \cap B, x \neq 0$. Then $a = \frac{x}{\|x\|} \in \chi(A) \cap \hat{B}$ and hence the function f_a belongs to the class $SP(\alpha)$. We have

$$\operatorname{Re} \left[e^{i\alpha} \frac{zf'_a(z)}{f_a(z)} \right] > 0, \quad z \in U.$$

From the equality

$$\frac{f'(za)(za)}{f(za)} = \frac{zf'_a(z)}{f_a(z)}, \quad z \in U,$$

we obtain

$$\operatorname{Re} \left[e^{i\alpha} \frac{f'(za)(za)}{f(za)} \right] > 0, \quad z \in U.$$

Putting $z = \|x\|$, we get (5).

Theorem 4 *Let $f \in H(B), f'(0) = A$ and $f(x) \neq 0$, for all $x \in B \setminus \{0\}$. If*

$$\operatorname{Re} \left[e^{i\alpha} \frac{f'(x)(x)}{f(x)} \right] > 0, \quad x \in B$$

then $f \in SP_A(\alpha)$.

Proof. Let $a \in \chi(A) \cap \hat{B}$. Since $f_a(0) = 0, f'_a(0) = 1, f_a(z) \neq 0$, for all $z \in U \setminus \{0\}$ and

$$\operatorname{Re} \left[e^{i\alpha} \frac{z f'_a(z)}{f_a(z)} \right] = \operatorname{Re} \left[e^{i\alpha} \frac{f'(za)(za)}{f(za)} \right] > 0, \quad z \in U,$$

we obtain that f_a is an α -spiral function in U . Then f is univalent in B_a and $f(B_a)$ is an α -spiral domain with respect to the origin. Hence $f \in SP_A(\alpha)$.

Remark

The above results can be generalized by replacing the unit ball B with a bounded and open set $D \subset E, D \neq \Phi$ such that $zD \subset D$, for $z \in \bar{U} = \{z \in \mathbb{C}, |z| \leq 1\}$. In this case, for $\alpha = 0$ some of the results due to E.Janiec [4] are obtained.

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